Probability Theory
2014/15 Semester IIb
Instructor: Daniel Valesin
Final Exam
16/6/2015
Duration: 3 hours

Name: ______________________________
Student number: ____________

This exam contains 10 pages (including this cover page) and 8 problems. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are allowed to have two hand-written sheets of paper and a calculator.

You are required to show your work on each problem except for Problem 1 (True or False).

Do not write on the table below.

Problem	Points	Score
1	10	
2	12	
3	12	
4	14	
5	12	
6	10	
7	10	
8	10	
Total:	90	

Standard Normal cumulative distribution function
The value given in the table is $F_X(x)$ for $X \sim \mathcal{N}(0,1)$.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

1. (10 points) In each of the items below, mark (T) if the statement is true and (F) if it is false. No justification is required.

a) (T) (F) If $X \sim \text{Bin}(n, p)$ and $c \in \{1, 2, \ldots\}$, then $X + c \sim \text{Bin}(n + c, p)$.

b) (T) (F) For any random variable X, $\mathbb{E}(X^2) \geq \mathbb{E}(X)^2$.

c) (T) (F) If f is the probability density function of a random variable X, then $0 \leq f(x) \leq 1$ for all $x \in \mathbb{R}$.

d) (T) (F) For a random variable X with finite variance, $\text{Var}(5X + 8) = 25\text{Var}(X) + 8$.

e) (T) (F) Any random variable is either discrete or continuous.

f) (T) (F) If (X, Y) follows a bivariate normal distribution, then X and Y are normally distributed.

g) (T) (F) For a random variable X with probability density function f_X, $\mathbb{E}(X^3) = 0$ if and only if $f_X(x) = f_X(-x)$ for all x

h) (T) (F) For two events A, B, we have $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ if and only if A and B are disjoint.

i) (T) (F) For any random variable X the following holds: $\mathbb{P}(X \geq \mathbb{E}(X)) = \frac{1}{2}$.

j) (T) (F) If A and B are independent events with $\mathbb{P}(A \cup B) = \frac{1}{3}$ and $\mathbb{P}(A) = \frac{1}{4}$, then $\mathbb{P}(A \cap B)$ is necessarily equal to $\frac{1}{36}$.

2. (a) (4 points) We roll a die 5 times. Find the probability that we obtain exactly three distinct numbers.

   (b) (4 points) n people, among them Alice, Bob, Charles and David are standing in line in an order that is chosen uniformly at random. Find the probability that Alice is next to Bob and Charles is next to David (that is, both things must occur simultaneously).

   (c) (4 points) In how many ways can we distribute 10 balls in 3 urns if the balls are indistinguishable and the urns are distinguishable?

3. (a) (4 points) If X is uniformly distributed on the interval $[0, 1]$ and $Y = e^X$, find the probability density function of Y.

   (b) (4 points) Prove: if X is a random variable taking values on $\{0, 1, 2, \ldots\}$, then for all $x \in \{1, 2, \ldots\}$ we have $\mathbb{P}(X \geq x) \leq \frac{\mathbb{E}(X)}{x}$.

   (c) (4 points) Prove using moment generating functions: if U and V are independent, $U \sim \text{Poisson}(\lambda_1)$ and $V \sim \text{Poisson}(\lambda_2)$, then $U + V \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

4. 50 men simultaneously throw their hats to the floor and then take hats at random from the floor.

   (a) (6 points) Find the expectation μ and variance σ^2 of the number of men who recover their own hat.

   (b) (4 points) Explain intuitively why the number of men who recover their own hat approximately follows a Poisson distribution with parameter 1.

   (c) (4 points) If the above experiment is repeated 100 times and $X_1, X_2, \ldots, X_{100}$ are the numbers of men who recover their own hats each time, estimate the probability that $\bar{X}_{100} = \frac{1}{100}\sum_{i=1}^{100} X_i$ is larger than $\mu + 0.08$.

5. The joint probability density function of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} C(x+2y) & \text{if } 0 < y < 1 \text{ and } 0 < x < 2; \\ 0 & \text{otherwise.} \end{cases}$$

   (a) (2 points) Find C.

   (b) (5 points) Find f_X.

   (c) (5 points) Find $f_{X,W}$, where $W = X + 2Y$.

6. (a) (5 points) Let X and Y be independent random variables with means μ_X and μ_Y and variances σ_X^2 and σ_Y^2. Prove that

$$\mathrm{Var}(XY) = \sigma_X^2\sigma_Y^2 + \mu_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2.$$

(b) (5 points) Let $X_1, \ldots, X_n$ be independent random variables with finite variance and let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. For a fixed $i \in \{1, \ldots, n\}$, prove that $\mathrm{Cov}(X_i - \bar{X}_n, \bar{X}_n) = 0$.

7. (a) (5 points) Let $U_1, U_2, \ldots$ be independent random variables uniformly distributed on the interval $[0,1]$ and $g : [0,1] \to \mathbb{R}$ be an integrable function. Show that $Y_n = \frac{1}{n}\sum_{i=1}^{n} g(U_i)$ converges in probability to a constant, and identify this constant.

   (b) (5 points) We draw a square of side length 2cm on the floor. Inscribed in the square, we draw a circle of radius 1cm (so that we can place the origin of the xy-plane in such a way that the vertices of the square are points $(-1,-1)$, $(-1,1)$, $(1,1)$, $(1,-1)$ and the circle has equation $x^2 + y^2 = 1$). We scatter 1000 tiny rocks on the square. The position of the rocks are independent and uniformly distributed on the square (this means that, if (X,Y) denotes the position of a rock, then the probability density function of (X,Y) is constant inside the square and zero outside it). Explain how this experiment can be used to approximate the value of π.

8. (10 points) Let $X_1, X_2, \ldots, X_{400}$ be independent and identically distributed random variables that take the value 1 with probability 1/10 and that take the value 10 with probability 9/10. Let $Y = X_1 \cdot X_2 \cdots X_{400}$. Use the Central Limit Theorem to approximate $\mathbb{P}(10^{354} \leq Y \leq 10^{363})$. *Hint: consider the logarithm of Y and write it as a sum of independent and identically distributed random variables.*